FALL 2024: MATH 790 EXAM 3

Throughout this exam, unless stated otherwise, all vector spaces will be defined over the field F. Each problem is worth 10 points. You will work in teams on this exam. You may use your notes, the Daily Summary, and any homework you have done (providing full details), but you may not consult any other sources, including, any algebra textbook, the internet, any graduate students not on your team, or any professor except your Math 790 instructor. You may not cite without proof any facts not covered in class or the homework. All members of each team should contribute to the team's effort. The solutions should be typeset in LaTex. Each team member should also participate in the typesetting effort. Each team should turn a hard copy of its solutions at the start of the final exam on Thursday, December 19 at 10:30am. **Good luck on the exam!**

1. Write V^* for the dual space of V, i.e., $\mathcal{L}(V,F)$. The elements of V^* are called *linear functionals* on V. Let $B := \{v_1,\dots,v_n\}$ be a basis for V.
   - (i) For each $1 \le j \le n$, define $v_j^* \in V^*$ by $v_j^*(v_i) = 1$, if $i = j$ and $v_j^*(v_i) = 0$, if $i \neq j$[1]. Show that $B^* := \{v_1^*,\dots,v_n^*\}$ is a basis for V^*. This basis is called the *dual basis* to B.
   - (ii) For $v \in V$, define $\hat{v} : V^* \to F$ by $\hat{v}(f) := f(v)$, for all $f \in V^*$. Prove that $\hat{v} \in (V^*)^*$ and $\hat{v_1},\dots,\hat{v_n}$ is a basis for $(V^*)^*$, the *double dual* of V.
   - (iii) Show directly, without using bases, that the map from $\phi : V \to (V^*)^*$ given by $\phi(v) := \hat{v}$ is an isomorphism of vector spaces. Because this map is a natural one, we say that V and $(V^*)^*$ are *canonically* isomorphic. In other words, this maps is independent of the choice of basis, where as V and V^* are non-canonically isomorphic, as any isomorphism between them arises by identifying bases.

Solution. For (i), let $f \in V^*$, and suppose $f(v_i) = \alpha_i$, for $1 \le i \le n$. Then $(\alpha_1 v_1^* + \cdots + \alpha_n v_n^*)(v_i) = \alpha_i$, for all i, so that $f = \sum_{i=1}^n \alpha_i v_i^*$. Thus, $\{v_i^*\}$ spans V^*. Suppose $\sum_i \beta_i v_i^* = 0$. Then, for any j, $0 = (\sum_i \beta_i v_i^*)(v_j) = \sum_i \beta_i v_i^*(v_j) = \beta_j$. Therefore, $\{v_i^*\}$ is a linearly independent set. Thus $\{v_i^*\}$ is a basis for V^*. Note: It follows that V and V^* have the same dimension.

For (ii), suppose $\sum_{i=1}^n \alpha_i \hat{v}_i = 0$. For any j we have

$$0 = (\sum_i \alpha_i \hat{v}_i)(v_j^*) = \sum_i \alpha_i \hat{v}_i(v_j^*) = \sum_i \alpha_i v_j^*(v_i) = \alpha_j.$$

This shows that the set $\{\hat{v_i}\}$ is linearly independent. Since $(V^*)^*$ has dimension n, the set $\{\hat{v_i}\}$ is a basis for $(V^*)^*$. For (iii), it is easy to check that ϕ is a linear transformation. Suppose $\phi(v) = 0$. Then $\hat{v} = 0$, so that $\hat{v}(f) = f(v) = 0$, for all $f \in V^*$. This implies $v = 0$, since otherwise, we could extend v to a basis of V and define $f : V \to F$ by sending v to 1 and all other basis elements to 0, which would be a contradiction. Thus ϕ is 1-1. Since V and $(V^*)^*$ have the same dimension, ϕ is an isomorphism. □

2. Maintaining the notation from the previous problem, suppose further that V is an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.
   - (i) For $v_0 \in V$ fixed, show that $\phi : V \to F$ given by $\phi(v) := \langle v, v_0\rangle$ belongs to V^*.
   - (ii) Conversely, given any $f \in V^*$, prove that there exists a *unique* $v_0 \in V$ such that $f(v) = \langle v, v_0\rangle$, for all $v \in V$. Hint: Work with an orthonormal basis for V.

Solution. Part (i) is immediate from the definition of inner product. For the converse, suppose $f \in V^*$. Fix an orthonormal basis $\{u_1,\dots,u_n\}$ for V, and suppose $f(u_i) = \alpha_i$, for all i. Set $v_0 := \sum_i \overline{\alpha_i} u_i$. Then for any j,

$$\langle u_j, v_0\rangle = \langle u_j, \sum_i \overline{\alpha_i} u_i\rangle = \sum_i \alpha_i \langle u_j, u_i\rangle = \alpha_j = f(u_j).$$

Thus, f and the linear functional $\langle -, v_0\rangle$ agree on a basis, so $f(v) = \langle v, v_0\rangle$, for all $v \in V$ □

3. Let $T \in \mathcal{L}(V,V)$. Suppose V is a finite dimensional inner product space over $\mathbb{C}$. This problem develops the standard definition of the adjoint of T. The goal is to construct a unique linear transformation $T^* \in \mathcal{L}(V,V)$ satisfying $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$, for all $v, w \in V$.
   - (i) Fix $w \in V$. Define $\phi_w : V \to F$ by $\phi_w(v) = \langle T(v), w\rangle$. Show that $\phi_w \in V^*$.

[1] Recall that to define a linear transformation with domain V, it suffices to specify its value on a basis

(ii) By The previous problem, there exists a unique $w_0 \in V$ such that $\phi_w(v) = \langle v, w_0\rangle$, for all $v \in V$. Set $T^*(w) := w_0$. Then, by definition, $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$, for all v. Doing this for each w gives a function $T^* : V \to V$. Show that $T^* \in \mathcal{L}(V, V)$.

(iii) Show that T^* is the unique element in $\mathcal{L}(V, V)$ satisfying $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$, for all $v \in V$.

(iv) Let B be an orthonormal basis for V. Show that $[T^*]_B^B = ([T]_B^B)^*$, so that the present definition of T^* agrees with the one given in class.

Solution. Part (i) follows immediately from the definitions of inner product and linear transformation. For part (ii), suppose $v_1, v_2 \in V$. Then for all $u \in V$ we have:

$$\langle u, T^*(v_1 + v_2)\rangle = \langle T(u), v_1 + v_2\rangle = \langle T(v), v_1\rangle + \langle T(u), v_2\rangle = \langle u, T^*(v_1)\rangle + \langle u, T^*(v_2)\rangle = \langle u, T^*(v_1) + T^*(v_2)\rangle.$$

Since this holds for all $u \in V$, $T^*(v_1 + v_2) = T^*(v_1) + T^*(v_2)$. The proof that $T^*(\lambda v) = \lambda T^*(v)$ for all $\lambda \in F$ and $v \in V$ is similar.

For (iii), suppose $S \in \mathcal{L}(V, V)$ satisfies $\langle T(v), w\rangle = \langle v, S(w)\rangle$, for all $v, w \in V$. Then $\langle v, T^*(w) - S(w)\rangle = 0$, for all v, w. Thus, $0 = T^*(w) - S(w)$, for all w, so that $T^* = S$.

4. A sequence $\mathcal{V}:\ \ 0 \xrightarrow{i} V_n \xrightarrow{T_n} V_{n-1} \xrightarrow{T_{n-1}} \cdots \xrightarrow{T_2} V_1 \xrightarrow{T_1} V_0 \xrightarrow{\pi} 0$ of vector spaces and linear transformations is a *complex* of vector spaces if the image of each T_{i+1} is contained in the kernel of T_i (including $i := T_{n+1}$ and $\pi := T_0$). The jth *homology* of the complex is the quotient space $\mathrm{H}_j(\mathcal{V}) := \ker(T_j)/\mathrm{im}(T_{j+1})$. Assume that each vector space has finite dimension. Show that $\Sigma_{j\geq 0}(-1)^{j+1}\dim(V_j) = \Sigma_{j\geq 0}(-1)^{j+1}\dim(\mathrm{H}_j(\mathcal{V}))$. Hint: Experiment with the $n = 2$ case.

Solution. We induct on n. Suppose $n = 1$. Then

$$\dim(V_0)-\dim(V_1) = \dim(V_0)-\{\dim\ker(T_1)+\dim\mathrm{im}(T_1)\} = \{\dim(V_0)-\dim\mathrm{im}(T_1)\}-\dim\ker(T_1) = \dim(H_0)-\dim(H_1).$$

Suppose $n > 1$. Applying the inductive hypothesis to the complex $0 \to V_{n-1} \xrightarrow{T_{n-1}} \cdots \xrightarrow{T_2} V_1 \xrightarrow{T_1} V_0 \xrightarrow{\pi} 0$, we have

$$\sum_{j=0}^{n-1}(-1)^{j+1}\dim(V_i) = \sum_{j=0}^{n-2}(-1)^{j+1}\dim(H_j) + (-1)^{n-1}\dim\ker(T_{n-1}).$$

Adding $(-1)^{n+1}\dim(V_n)$ to both sides gives

$$\begin{aligned}\sum_{j=0}^{n}(-1)^{j+1}\dim(V_i) &= \sum_{j=0}^{n-2}(-1)^{j+1}\dim(H_j) + (-1)^{n}\dim\ker(T_{n-1}) + (-1)^{n+1}\dim(V_n)\\ &= \sum_{j=0}^{n-2}(-1)^{j+1}\dim(H_j) + (-1)^{n}\dim\ker(T_{n-1}) + (-1)^{n+1}\dim\mathrm{im}(T_n) + (-1)^{n+1}\dim\ker(T_n)\\ &= \sum_{j=0}^{n-2}(-1)^{j+1}\dim(H_j) + (-1)^{n}\dim(H_{n-1}) + \dim\ker(T_n)\}\\ &= \sum_{j=1}^{n}(-1)^{j+1}\dim(H_j). \quad \square\end{aligned}$$

5. Set $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{M}_n(\mathbb{R})$.

(i) For $p \geq 1$, find p distinct pth roots of A

(ii) Find the solution to the system of first order linear differential equations given by the vector equation $\mathbf{X}'(t) = A \cdot \mathbf{X}(t)$, with initial condition $\mathbf{X}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Here $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

Solution. For (i), we can easily calculate that 1, -1 are the eigenvalues of A, for $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $P^{-1} = \frac{1}{2}\cdot\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, with $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If we set $C := \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{p+1} \end{pmatrix}$, then $C^p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P^{-1}AP$. It follows that for $B := PCP^{-1}$, $B^p = A$. Note $B = \frac{1}{2}\cdot\begin{pmatrix} 1+(-1)^{p+1} & 1+(-1)^{p} \\ -1+(-1)^{p+1} & -1+(-1)^{p} \end{pmatrix}$. If we set $\epsilon = e^{\frac{2\pi i}{p}}$, then $B, \epsilon B, \ldots, \epsilon^{p-1}B$ are p distinct pth roots of A.

For (ii), we have from our Daily Update of November 11, the solution to the system of equations is given by $e^{At} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, so we need to calculate e^{At}. Using what we have in part (i), For $Dt = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, $e^{Dt} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Thus,

$$e^{At} = Qe^{Dt}Q^{-1} = \frac{1}{2} \cdot \begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}.$$

6. Let $\{V_i\}_{i \in I}$ be a collection of (not necessarily finite dimensional) vector spaces. Prove the following properties of the direct sum $\bigoplus_{i \in I} V_i$.

(a) For each $i_0 \in I$ there is a natural inclusion of vector spaces $j_{i_0} : V_{i_0} \to \bigoplus_{i \in I} V_i$.

(b) The direct sum $\bigoplus_{i \in I} V_i$ satisfies the following universal property: Given a vector space U and a collection of linear transformations $f_i : V_i \to U$, for each $i \in I$, there exists a unique linear transformation $f : \bigoplus_{i \in I} V_i \to U$ satisfying $f \circ j_i = f_i$, for all $i \in I$.

(c) For any vector space U, $(\bigoplus_{i \in I} V_i) \otimes U \cong \bigoplus_{i \in I}(V_i \otimes U)$.

Solution. Part (a) is straight forward - for any $i_o \in I$, we define $\jmath_{i_0} : V_{i_0} \to \bigoplus V_i$ to be that map which takes any $x \in V_{i_0}$ to the I-tuple that is zero in every coordinate, except the i_0 coordinate, and is equal to x in the i_0 coordinate.$\jmath_{i_0}$ is clearly an injective linear transformation.

For (b) Suppose we are given a vector space U and a collection of linear transformations $f_i : V_i \to U$, for each $i \in I$. Define $f : \bigoplus_i V_i \to U$ as follows: For $t := (v_i)_{i \in I} \in \bigoplus_i V_i$, $f(t) := \sum_i f_i(v_i)$. This sum makes since, since only finitely many of the coordinates of t are non-zero. Note that f is well defined, since the representation of elements in the direct sum is unique. Also, it is easy to check that f is a linear transformation; it clearly satisfies $f \circ j_i = f_i$, for all $i \in I$.

For part (c), we with a diagram

$$\begin{array}{ccc} (\bigoplus_i V_i) \times U & \xrightarrow{f} & (\bigoplus_i V_i) \otimes U \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle T} & \\ \bigoplus_i (V_i \otimes U), & & \end{array}$$

where f is the canonical map associated to the tensor product and g is the bilinear (easy to check) map defined by $g((v_i)_{i \in I}, u) := (v_i \otimes u)_{i \in I}$. Then there exists a unique linear transformation $T : (\bigoplus_i V_i) \otimes U \to \bigoplus_i (V_i \otimes U)$ satisfying $T((v_i)_i \otimes u) = (v_i \otimes u)_i$, for all $v_i \in V_i$ and $u \in U$. To obtain a map in the opposite direct, we start with the diagram

$$\begin{array}{ccc} V_i \times U & \xrightarrow{f'} & V_i \otimes U \\ {\scriptstyle g_i'}\downarrow & \swarrow {\scriptstyle T_i'} & \\ (\bigoplus_i V_i) \otimes U & & \end{array}$$

with $g_i' : V_i \times U \to (\bigoplus_i V_i) \otimes U$, the map defined by $g_i'(v_i, u) = j_i(v_i) \otimes u$. Then, there exists a unique $T_i' : V_i \otimes U \to (\bigoplus_i V_i) \otimes U$ satisfying $T_i'(v_i \otimes u) = j_i(v_i) \otimes u$. Maintaining the notation from part (b), we now have a unique linear transformation $S : \bigoplus_i (V_i \otimes U) \to (\bigoplus_i V_i) \otimes U$ satisfying $S \circ j_i = T_i'$, for all $i \in I$. We check that $S \circ T$ is the identity map, and leave it to you to check that $T \circ S$ is the identity map, showing that $(\bigoplus_{i \in I} V_i) \otimes U \cong \bigoplus_{i \in I}(V_i \otimes U)$. For $(v_i)_i \otimes u \in (\bigoplus_i V_i) \otimes U$, we have

$$ST((v_i)_i \otimes u) = S((v_i \otimes u)_i) = \sum_i T_i'(v_i \otimes u) = \sum_i j_i(v_i) \otimes u = (v_i)_i \otimes u.$$

□

7. Let V, W, U be vector spaces that need not be finite dimensional. Prove that $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$.

Solution. We need two linear transformations $T : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ and $S : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ such that ST is the identity transformation on $U \otimes (V \otimes W)$ and TS is the identity transformation on $(U \otimes V) \otimes W$. We will obtain T if we can find a bilinear transformation $\psi : U \times (V \otimes W) \to (U \otimes V) \otimes W$. For this, we need a map from $V \otimes W$ to $(U \otimes V) \otimes W$. We temporarily fix $u \in U$ and define $\psi_u : V \times W \to (U \otimes V) \otimes W$ by $\psi_u(v, w) := (u \otimes v) \otimes w$. It is straightforward to check that ϕ_u is bilinear. For example,

$$\phi_u(v_1 + v_2, w) = (u \otimes (v_1 + v_2)) \otimes w = (u \otimes v_1 + u \otimes v_2) \otimes w = (u_1 \otimes v_1) \otimes w + (u \otimes v_2) \otimes w = \psi_u(v_1, w) + \psi_u(v_2, w).$$

The other conditions for the bilinearity of ϕ_u follow in a similar manner. Thus, there exists a linear transformation $T_u : V \otimes W \to (U \otimes V) \otimes W$ such that $T_u \circ \phi_0 = \psi_u$, where $\phi_0 : U \times V \otimes W \to U \otimes (V \otimes W)$ is the canonical map taking (u, t) to $u \otimes t$, for all $t \in V \otimes W$. In particular, $T_u(v \otimes w) = (u \otimes v) \otimes w$, for all $v \otimes w \in V \otimes W$.

Now, define $\psi : U \times (V \otimes W)$ by $\psi(u, t) = T_u(t)$, for all $t \in V \otimes W$. It is easy to check that ψ is well defined, and thus there exists $T : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ such that $T \circ \phi = \psi$, where $\phi : U \times (V \otimes W) \to U \otimes (V \otimes W)$ is the canonical map taking (u, ρ) to $u \otimes \rho$, for all $\rho \in V \otimes W$. In particular, $T(u \otimes (v \otimes w)) = (u \otimes v) \otimes w$, for all $u \otimes (v \otimes w) \in U \otimes (V \otimes W)$.

In exactly the same way, we obtain a linear transformation $S : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ satisfying $S((u \otimes v) \otimes w) = u \otimes (v \otimes w)$, for all $(u \otimes v) \otimes w$ in $(U \otimes V) \otimes W$. Since $ST(u \otimes (v \otimes w)) = u \otimes (v \otimes w)$ for all $u \otimes (v \otimes w) \in U \otimes (V \otimes W)$, it follows that ST is the identity transformation on $U \otimes (V \otimes W)$, since these elements span $U \otimes (V \otimes W)$. Similarly, TS is the identity transformation on $(U \otimes V) \otimes W$, and the proof is complete.

8. The vector spaces below need not be finite dimensional.

(i) Suppose that $T : L \to M$ and $S : M \to L$ are linear transformations of vector spaces such that ST is the identity on L and TS is the identity on M. Prove that T is an isomorphism with inverse S.

(Ii) Suppose (P, f) is a tensor product of V and W. Suppose $\alpha : P \to P_1$ is an isomorphism of vector spaces. Set $f_1 := \alpha \circ f$. Show that (P_1, f_1) is a tensor product of V and W.

Solution. For (i), the second statement follows immediately from the first statement. Suppose $T(x) = 0$, for $x \in L$. Then $x = ST(x) = S(0) = 0$, so the kernel of T equals zero, and thus, T is -1-. Take $y \in M$ and set $x := S(y)$. Then $y = TS(y) = T(x)$, showing that T is surjective, and thus, an isomorphism.

For (ii), we first note that $\alpha \circ f : V \times W \to P'$ is bilinear. For example,

$$\alpha \circ f(v_1 + v_2, w) = \alpha(f(v_1 + v_2, w)) = \alpha(f(v_1, w) + f(v_2, w)) = \alpha(f(v_1, w)) + \alpha(f(v_2, w)) = \alpha \circ f(v_1, w) + \alpha \circ f(v_2, w).$$

The other bilinear properties follow in a similar fashion. Now suppose we are given a bilinear map $g : V \times W \to U$. Then there exists a unique linear transformation $T : P \to U$ such that $T \circ f = g$. Thus, we have $T \circ \alpha^{-1} : P' \to U$. Note that

$$(T \circ \alpha^{-1}) \circ f_1 = (T \circ \alpha^{-1})(\alpha \circ f) = T \circ f = g.$$

The proof is complete if we show that $T \circ \alpha^{-1}$ is unique. Suppose $S : P' \to U$ satisfies $S \circ f_1 = g$. Then $S \circ (\alpha \circ f) = g$. Thus, $(S \circ \alpha) \circ f = g$. Thus, since (P, f) is a tensor product, $S \circ \alpha = T$. Therefore, $S = T \circ \alpha^{-1}$, which is what we want. □

9. Let V be a vector space over F. Let L denote $\text{Span}\{v \otimes v' - v' \otimes v \mid v, v' \in V\} \subseteq V \otimes V$. Let $v_1 * v_2$ denote the coset $v_1 \otimes v_2 + L$ in the quotient space $(V \otimes V)/L$. Set $S^2(V) := (V \otimes V)/L$, the *symmetric square* of V.

(i) Show that the same bilinear properties holding in $V \otimes V$ hold with respect to the product $*$ in $S^2(V)$.

(ii) Show that $v_1 * v_2 = v_2 * v_1$ in $S^2(V)$, for all $v_1, v_2 \in V$.

(iii) Given a vector space U, a bilinear map $h : V \times V \to U$ is *symmetric* if $h(v_1, v_2) = h(v_2, v_1)$ for all $v_1, v_2 \in V$. Let $\hat{f} : V \times V \to S^2(V)$ be the natural map i.e., the usual bilinear map $f : V \times V \to V \otimes V$ followed by the quotient map from $V \otimes V \to S^2(V)$. Prove that $\hat{f}$ is a symmetric bilinear map, and given any vector space U and a symmetric bilinear map $g : V \times V \to U$, there exists a unique linear transformation $T : S^2(V) \to U$ such that $T \circ \hat{f} = g$

(iv) Suppose $v_1, \ldots, v_n$ is a basis for V. Find a basis for $S^2(V)$.

(v) If $\dim(V) = n$, what is $\dim(S^2(V))$?

Solution. For (i), we have

$$(v_1 + v_2) * w = (v_1 + v_2) \otimes w + L = \{v_1 \otimes w + v_2 \otimes w\} + L = \{(v_1 \otimes w) + L\} + \{(v_2 \otimes w) + L\} = v_1 * w + v_2 * w.$$

The other bilinear properties follow in a similar fashion.

For (ii), we have $v_1 \otimes v_2 - v_2 \otimes v_1 \in L$, so that $v_1 \otimes v_2 + L = v_2 \otimes v_2 + L$, and by definition, $v_1 * v_2 = v_2 * v_1$.

For (iii), the proof that $\hat{f}$ is symmetric is similar to the proof in part (ii). Now suppose $g : V \times V \to U$ is bilinear and symmetric. Then, Since g is bilinear, we have a unique map $T_0 : V \otimes V \to U$ such that $T_0 \circ \hat{f} = g$. On the other hand, for $v_1, v_2 \in V$, $T_0(v_1 \otimes v_2 - v_2 \otimes v_1) = T_0 f(v_1, v_2)) - T_0 f(v_2, v_2) = g(v_1, v_2) - g(v_2, v_1) = 0$, since g is symmetric. Thus, T_0 is zero on the elements of L, and thus L is in the kernel of T_0. It follows that there is an induced map $T : V \otimes V/L = S^2(V) \to U$ such that $T(v_1 * v_2) = T_0(v_1 \otimes v_2)$, and thus, $T \circ \hat{f} = g$. One can easily check that T is unique, in standard fashion.

For (iv), since the n^2 elements $\{v_i \otimes v_j\}_{1 \leq i,j \leq n}$ form a basis for $V \otimes V$, their images $\{v_i * v_j\}_{1 \leq i,j \leq n}$ span $S^2(V)$. Thus, their is a linearly independent subset of this set that forms a basis for $S^2(V)$. Of course, for $i \neq j$, $v_i * v_j = v_j * v_i$. We will show that getting rid of the repetitions leads to a basis. For this we delete $v_i * * v_j$ with $i > j$. This the same set of vectors, but without repetitions, i.e., $B := \{v_i * v_i\}_{1\le i\le j\le n}$ still spans $S^2(V)$. To show that the set B is linearly independent, consider the polynomial ring in the n variables $x_1, \dots, x_n$. The set U of all homogenous polynomials of degree two forms a vector space of dimension $\frac{n(n+1)}{2}$, whose basis consists of all monomials of degree two, namely, $\{x_i x_j\}_{1\le i,j\le n}$. Note that this set also has $\frac{n(n+1)}{2}$ elements. Define $g : V \times V \to U$ as follows: For $u = \sum_{i=1}^n a_i v_i$ and $w = \sum_{j=1}^n b_j v_j$ in V, $g(u,v) = (\sum_i a_i x_i) \cdot (\sum_j b_j x_j)$. Since multiplication of polynomials is bilnear (by the distributive property) and commutative, g is a bilinear, symmetric map. Thus, there exists a (unique) $T : V * V \to U$ such that $T \circ \hat{f} = g$. Now suppose $\sum_{1\le i\le j\le n} c_{ij}(v_i * v_j) = 0$. In other words, $\sum_{1\le i\le j\le n} c_{ij}\hat{f}(v_i, v_j) = 0$. Applying T we get

$$\sum_{1\le i\le j\le n} c_{ij} T\hat{f}(v_i, v_j) = \sum_{1\le i\le j\le n} c_{ij} g(v_i, v_j) = \sum_{1\le i\le j\le n} c_{ij} x^i y^j = 0.$$

Thus, each $c_{ij} = 0$, so that B is a basis for $V * V$.

By part (iv), the dimension of $V * V = \frac{n(n+1)}{2}$. □

10. Let V and W be vector spaces. Recall that V^* denotes the dual space of V.
    (i) Prove that there exists a unique linear transformation $T : V^* \otimes W \to \mathcal{L}(V, W)$ such that $T(f \otimes w)(v) = f(v)w$, for all $f \otimes w \in V^* \otimes W$ and $v \in V$.
    (ii) Prove that if V and W are finite dimensional, then T is an isomorphism.
    (iii) Let $V = \mathbb{R}^4$, $W = \mathbb{R}^3$, $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \in V^*$, $f_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} \in V^*$, $w_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Find the matrix of $T(f_1 \otimes w_1 + f_2 \otimes w_2)$ with respect to the standard bases of V and W.

Solution. Solution. For part (i), let $\phi : V^* \times W \to V^* \otimes W$ be the given map. We define $h : V^* \times W \to \mathcal{L}(V, W)$ as $h(f, w) = f_w$, where for all $v \in V$, $f_w(v) := f(v)w$. Since f is linear, it follows immediately that $f_w \in \mathcal{L}(V, W)$, for all $w \in W$ and $f \in V^*$. Now, for $f_1, f_2 \in V^*$ and $w \in W$, $h(f_1 + f_2, w) = (f_1 + f_2)_w$. Thus, for all $v \in V$ we have,

$$\begin{aligned} h(f_1 + f_2, w)(v) &= (f_1 + f_2)(v)w \\ &= (f_1(v) + f_2(v))w \\ &= f_1(v)w + f_2(v)w \\ &= h(f_1, w)(v) + h(f_2, w)(v), \end{aligned}$$

and thus, $h(f_1 + f_2, w) = h(f_1, w) + h(f_2, w)$. Similarly, one can show $h(f, w_1 + w_2) = h(f, w_1) + h(f, w_2)$. For $\lambda \in F$,

$$h(\lambda f, w)(v) = (\lambda f)(v)w = \lambda\{f(v)w\} = \lambda\{h(f, w)(v)\},$$

showing that $h(\lambda f, w) = \lambda h(f, w)$. Similarly, one can show $h(f, \lambda w) = \lambda h(f, w)$, so that h is bilinear. Thus, there exists a unique linear transformation $T : V^* \otimes W \to \mathcal{L}(V, W)$ so that $T\phi = h$. In other words, $T(f \otimes w) = f_w$, which means $T(f \otimes w)(v) = f_w(v) = f(v)w$, for all $v \in V$.

For part (ii), suppose that $\{v_1, \dots, v_n\}$ is a basis for V and $\{w_1, \dots, w_m\}$ is a basis for W. Let $\{v_1^*, \dots, v_n^*\}$ denote the corresponding dual basis for V^*. Then $\{v_i^* \otimes w_j\}_{1\le i\le n, 1\le j\le m}$ is a basis for $V^* \otimes W$. Now, for each v_k, $T(v_i^* \otimes w_j)(v_k) = w_j$, if $k = i$ and $T(v_i^* \otimes w_j)(v_k) = 0$, if $k \ne i$. This shows that $T(v_i^* \otimes w_j)$ is a basis for $\mathcal{L}(V, W)$. In other words, T takes a basis of $V^* \otimes W$ to a basis of $\mathcal{L}(V, W)$, and hence these spaces are isomorphic.

For part (iii), let $E := \{e_1, e_2, e_3, e_4\}$ denote the standard basis of $\mathbb{R}^4$ and U denote the standard basis of $\mathbb{R}^3$. Then,

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_1) = T(f_1 \otimes w_1)(e_1) + T(f_2 \otimes w_2)(e_2) = 1 \cdot w_1 + -1 \cdot w_2 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_2) = T(f_1 \otimes w_1)(e_2) + T(f_2 \otimes w_2)(e_2) = 2 \cdot w_1 + 0 \cdot w_2 = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}.$$

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_3) = T(f_1 \otimes w_1)(e_3) + T(f_2 \otimes w_2)(e_3) = 3 \cdot w_1 + 1 \cdot w_2 = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}.$$

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_4) = T(f_1 \otimes w_1)(e_4) + T(f_2 \otimes w_2)(e_4) = 4 \cdot w_1 + -0 \cdot w_2 = \begin{pmatrix} 12 \\ 8 \\ 4 \end{pmatrix}.$$

Therefore,

$$[T(f_1 \otimes w_1 + f_2 \otimes w_2]_E^U = \begin{pmatrix} 4 & 6 & 8 & 12 \\ 2 & 4 & 6 & 8 \\ 0 & 2 & 4 & 4 \end{pmatrix}.$$

Bonus Problems. Each problem below is worth 10 points. Solutions must be completely correct in order to receive any credit.

(i) Let $\{f_n\}$ be the Fibonacci sequence $f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ ....,\ f_n = f_{n-1} + f_{n-2}$. Prove that for all $n \geq 1$,

$$f_n = \frac{1}{\sqrt{5}}\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\}.$$

Hint: Write $\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = A^{n-2} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$, for some 2×2 matrix A and $u, v \in \mathbb{Z}$.

(ii) Let A and B be $n \times n$ matrices and set $C := AB - BA$. If $AC = CA$, prove that C is a nilpotent matrix.

(iii) Let V and W be vector spaces over $\mathbb{C}$ of dimensions n and m. Set $U := \mathcal{L}(V, W)$. Fix isomorphisms $\alpha \in \mathcal{L}(V, V)$ and $\beta \in \mathcal{L}(W, W)$. Define $\phi \in \mathcal{L}(U, U)$ by $\phi(T) = \beta^{-1}T\alpha$, for all $T \in U$. Find formulas for trace(ϕ) and det(ϕ) in terms of α and β.